New Examples of Potential Theory on Bratteli Diagrams

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Abstract

We consider potential theory on Bratteli diagrams arising from Macdonald polynomials. The case of Hall-Littlewood polynomials is particularly interesting; the elements of the diagram are partitions, the branching multiplicites are integers, the combinatorial dimensions are Green's polynomials, and the Jordan form of a randomly chosen unipotent upper triangular matrix over a finite field gives rise to a harmonic function. The case of Schur functions yields natural deformations of the Young lattice and Plancharel measure. Many harmonic functions are constructed and algorithms for sampling from the underlying probability measures are given.

1 Introduction

Potential theory on Bratteli diagrams is a beautiful subject, with connections to probability and representation theory. The basic set-up is as follows (for more details see Kerov's lovely article [Ke1]). One starts with a Bratteli diagram; that is an oriented graded graph $\Gamma = \cup_{n \geq 0} \Gamma_n$ such that

1. Γ_0 is a single vertex $\emptyset$.

2. If the starting vertex of an edge is in Γ_i, then its end vertex is in Γ_{i+1}.

3. Every vertex has at least one outgoing edge.

4. All Γ_i are finite.

For two vertices $\lambda, \Lambda \in \Gamma$, one writes $\lambda \nearrow \Lambda$ if there is an edge from λ to Λ. Part of the underlying data is a multiplicity function $\kappa(\lambda, \Lambda)$. Letting the weight of a path in Γ be the product of the multiplicities of its edges, one defines the dimension $dim(\Lambda)$ of a vertex Λ to be the sum of the weights over all maximal length paths from $\emptyset$ to Λ (this definition clearly extend to intervals). An important concept, which we will be defined carefully in Section 3, is the boundary of a branching.

Given a Bratteli diagram with a multiplicity function, one calls a function ϕ *harmonic* if $\phi(0) = 1$, $\phi(\lambda) \geq 0$ for all $\lambda \in \Gamma$, and

$$\phi(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} \kappa(\lambda, \Lambda) \phi(\Lambda).$$

An equivalent concept is that of coherent probability distributions. Namely a set $\{M_n\}$ of probability distributions M_n on Γ_n is called *coherent* if

$$M_{n-1}(\lambda) = \sum_{\Lambda : \lambda \nearrow \Lambda} \frac{dim(\lambda)\kappa(\lambda, \Lambda)}{dim(\Lambda)} M_n(\Lambda).$$

The formula allowing one to move between the definitions is $\phi(\lambda) = \frac{M_n(\lambda)}{dim(\lambda)}$.

One reason the set-up is interesting from the viewpoint of probability theory is the fact that every harmonic function can be written as a Poisson integral over the set of extreme harmonic functions (which is often the Martin boundary). For the Pascal lattice (vertices of Γ_n are pairs (k, n) with $k = 0, 1, \cdots, n$ and (k, n) is connected to $(k, n + 1)$ and $(k + 1, n + 1)$), this fact is the simplest instance of de Finetti's theorem. When the multiplicity function κ is integer valued, one can define a sequence of algebras A_n associated to the Bratteli diagram, and harmonic functions correspond to certain characters of the inductive limit of the algebras A_n.

Several examples of the above constuctions have been examined in detail. These include characters of the infinite symmetric group [KeV], Kingman's branching (related to population genetics and to a deformation of the uniform measure on the symmetric group) [Kin], Jack branching (which

generalizes the previous two examples and is also related to spherical functions of the infinite hyperoctahedral Gelfand pair) [KOO], and differential posets [GK]. The paper [BO] gives an update of recent developments, and the book [GDJ] contains much of interest.

The point of this note is to provide new examples of potential theory on Bratteli diagrams. The most interesting such example arises from the probabilistic study of the Jordan form of a uniformly chosen element of $T(n)$, the group of upper triangular matrices over a finite field, with 1's along the main diagonal. Although Hall-Littlewood polynomials come into play, the underlying Bratteli diagram is different from the Hall-Littlewood branching defined in [Ke2]. In particular, the multiplicty function is integer valued. The Bratteli diagrams examined here arise from work of Garsia and Haiman [GH] on Macdonald polynomials; to the best of our knowledge this is the first attempt to examine them from the viewpoint of potential theory.

The computation of the Martin boundary for these branchings is a hard open problem. The two main methods for computing boundaries are the method of positive homomorphisms and the ergodic method [Ke1]. It is unclear whether the Bratteli diagrams studied here have multiplicative branching, which blocks use of the first method. The ergodic method relies on precise estimates for ratios of dimensions in the Bratteli diagram; at present this is blocked by the current combinatorial intractability of Kostka-Foulkes polynomials. Nevertheless, in one simple case, that of Schur functions, the ergodic method does lead to a determination of the Martin boundary.

The sampling algorithms given here (and indeed this whole note) were motivated by an effort of the author [F2] to understand a probabilistic growth algorithm of Borodin [B] and Kirillov [Ki1] for Jordan form of uniformly chosen elements of $T(n)$ in terms of symmetric function theory. For the reader's benefit, we remark that the q, t hook walk in [GH] (defined on Young tableaux of a given shape) is different from the probabilistic growth algorithms given here; the paper [GNW] and some of [Ke3] however are specializations of our sampling algorithms. Section 2 gives the general construction, and Section 3 gives examples.

2   The General Construction

To begin we introduce some notation, as on pages 2-5 of [M]. Let λ be a partition of a non-negative integer $n = \sum_i \lambda_i$ into non-negative integral parts $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$. The notation $\lambda \vdash n$ or $|\lambda| = n$ will mean that λ is a partition of n. Let $m_i(\lambda)$ be the number of parts of λ of size i, and let λ' be the partition dual to λ in the sense that $\lambda'_i = m_i(\lambda) + m_{i+1}(\lambda) + \cdots$. Let $n(\lambda)$ be the quantity $\sum_{i \geq 1}(i-1)\lambda_i$. It is also useful to define the diagram associated to λ as the set of points $(i, j) \in Z^2$ such that $1 \leq j \leq \lambda_i$. We use the convention that the row index i increases as one goes downward and the column index j increases as one goes across. So the diagram of the partition (5441) is:

$$
\begin{array}{ccccc}
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & & & &
\end{array}
$$

For $\lambda \nearrow \Lambda$, let $R_{\Lambda/\lambda}$ (resp. $C_{\Lambda/\lambda}$) be the squares of λ in the same row (resp. colmun) as the square removed from λ to get Λ. This notation differs from that in [M]. Let $a_\lambda(s)$, $l_\lambda(s)$ be the number of cells in λ strictly to the east and south of s, and let $h_\lambda(s) = a_\lambda(s) + l_\lambda(s) + 1$. The notation $[n]$ will mean $\frac{q^n - 1}{q - 1}$, the q-analog of the number n. The symbol $\psi'_{\Lambda/\lambda}$ (as on page 341 of [M]) denotes

$$
\prod_{s \in C_{\Lambda/\lambda}} \frac{1 - q^{a_\Lambda(s)} t^{l_\Lambda(s)+1}}{1 - q^{a_\Lambda(s)+1} t^{l_\Lambda(s)}} \frac{1 - q^{a_\lambda(s)+1} t^{l_\lambda(s)}}{1 - q^{a_\lambda(s)} t^{l_\lambda(s)+1}}.
$$

For $\lambda \vdash n$, f^λ will denote the dimension of the irreducible representation of the symmetric group S_n parameterized by λ. Let $K_{\mu\Lambda}$ be the Kostka-Foulkes polynomial, as in Section 6.8 of [M] and let $P_\Lambda(q,t)$ be Macdonald's polynomial.

Definition 1: For $0 \le q < 1$ and $0 < t < 1$, the underlying Bratteli diagram Γ has as level Γ_n all partitions λ of n. For $\lambda \nearrow \Lambda$, the multiplicty function is defined as

$$\kappa(\lambda, \Lambda) = \prod_{s \in R_{\Lambda/\lambda}} \frac{t^{-l_\Lambda(s)} - q^{a_\Lambda(s)+1}}{t^{-l_\lambda(s)} - q^{a_\lambda(s)+1}} \prod_{s \in C_{\Lambda/\lambda}} \frac{q^{a_\Lambda(s)} - t^{-l_\Lambda(s)+1}}{q^{a_\lambda(s)} - t^{-l_\lambda(s)+1}}.$$

Letting i be the column number of the square removed to go from λ to Λ, this can be rewritten as

$$\frac{1}{t^{\Lambda_i'-1}} \prod_{s \in R_{\Lambda/\lambda}} \frac{1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)}}{1 - q^{a_\lambda(s)+1}t^{l_\lambda(s)}} \prod_{s \in C_{\Lambda/\lambda}} \frac{1 - q^{a_\Lambda(s)}t^{l_\Lambda(s)+1}}{1 - q^{a_\lambda(s)}t^{l_\lambda(s)+1}}.$$

Equation I.10 of [GH] proves that

$$dim(\Lambda) = \frac{1}{t^{n(\Lambda)}} \sum_{\mu \vdash n} f^\mu K_{\mu\Lambda}(q,t).$$

Definition 2: For $0 \le q < 1, 0 < t < 1$ and $0 \le x_1, x_2, \cdots$ such that $\sum x_i = 1$, define a family $\{M_n\}$ of probability measures on partitions of size n by

$$
\begin{aligned}
M_n(\Lambda) &= \frac{(1-q)^{|\Lambda|} P_\Lambda(x; q, t) \sum_{\mu \vdash n} f^\mu K_{\mu\Lambda}(q,t)}{\prod_{s \in \Lambda}(1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)})} \\
&= \frac{(1-q)^{|\Lambda|} t^{n(\Lambda)} dim(\Lambda)}{\prod_{s \in \Lambda}(1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)})}
\end{aligned}
$$

It will soon be verified later that the $M_n(\lambda)$ are in fact probability measures, and also that they are coherent with respect to the diagram of Definition 1.

Lemma 1 gives the combinatorial analog of the probabilistic notion of exchangeability.

Lemma 1 *Let $\gamma(0) = \emptyset \nearrow \gamma(1) \cdots \nearrow \gamma(n) = \Lambda$ be any path in the Bratteli diagram. Then*

$$\prod_{j=1}^n \frac{\psi'_{\gamma(j)/\gamma(j-1)}}{\kappa(\gamma(j-1), \gamma(j))} = \frac{(1-q)^n t^{n(\Lambda)}}{\prod_{s \in \Lambda}(1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)})}.$$

In paticular, the product depends on the path only through its endpoint.

PROOF: Suppose that $\gamma(j)$ is obtained from $\gamma(j-1)$ by adding to column i. Writing everything out, one sees that

$$
\begin{aligned}
\frac{\psi'_{\gamma(j)/\gamma(j-1)}}{\kappa(\gamma(j-1), \gamma(j))} &= t^{\gamma(j)_i'-1} \prod_{s \in C_{\Lambda/\lambda} \cup R_{\Lambda/\lambda}} \frac{1 - q^{a_\lambda(s)+1}t^{l_\lambda(s)}}{1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)}} \\
&= t^{\gamma(j)_i'-1}(1-q) \frac{\prod_{s \in \lambda} 1 - q^{a_\lambda(s)+1}t^{l_\lambda(s)}}{\prod_{s \in \Lambda} 1 - q^{a_\Lambda(s)+1}t^{l_\Lambda(s)}}.
\end{aligned}
$$

Using the fact that $n(\Lambda) = \sum_{j=1}^n (\gamma(j)_i' - 1)$ and multiplying terms, the result follows. $\square$

Theorem 1 proves that the family $\{M_n\}$ satisfies the coherence equation. It will then be seen that the $\{M_n\}$ are indeed probability measures.

4

Theorem 1 *For any $0 \leq q < 1$ and $0 < t < 1$ and $0 \leq x_1, x_2, \cdots$ satisfying $\sum x_i = 1$, the set $\{M_n\}$ satisfy the equation*

$$M_{n-1}(\lambda) = \sum_{\Lambda:\lambda \nearrow \Lambda} \frac{dim(\lambda)}{dim(\Lambda)} \kappa(\lambda, \Lambda) M_n(\Lambda).$$

PROOF: By Lemma 1 and the definition of $M_n(\lambda)$, any path from $\emptyset$ to Λ yields the equality

$$M_n(\Lambda) = P_\Lambda dim(\Lambda) \prod_{j=1}^{n} \left(\frac{\psi'_{\gamma(j)/\gamma(j-1)}}{\kappa(\gamma(j-1), \gamma(j))} \right).$$

In the following equations, paths from $\emptyset$ to some Λ such that $\lambda \nearrow \Lambda$ are chosen so as to first go to λ (in a way independent of Λ) and then go to Λ. Consequently,

$$
\begin{aligned}
&\sum_{\Lambda:\lambda \nearrow \Lambda} \frac{dim(\lambda)}{dim(\Lambda)} \kappa(\lambda, \Lambda) M_n(\Lambda) \\
=~ &\sum_{\Lambda:\lambda \nearrow \Lambda} dim(\lambda) \kappa(\lambda, \Lambda) P_\Lambda \prod_{j=1}^{n-1} \left(\frac{\psi'_{\gamma(j)/\gamma(j-1)}}{\kappa(\gamma(j-1), \gamma(j))} \right) \frac{\psi'_{\Lambda/\lambda}}{\kappa(\lambda, \Lambda)} \\
=~ &dim(\lambda) P_\lambda \prod_{j=1}^{n-1} \left(\frac{\psi'_{\gamma(j)/\gamma(j-1)}}{\kappa(\gamma(j-1), \gamma(j))} \right) \sum_{\Lambda:\lambda \nearrow \Lambda} \frac{P_\Lambda \psi'_{\Lambda/\lambda}}{P_\lambda} \\
=~ &M_{n-1}(\lambda) \sum_{\Lambda:\lambda \nearrow \Lambda} \frac{P_\Lambda \psi'_{\Lambda/\lambda}}{P_\lambda} \\
=~ &M_{n-1}(\lambda)
\end{aligned}
$$

Since $\sum x_i = 1$, the final equality is simply equation 6.24 on page 340 of [M] with $r = 1$ (a Pieri rule). $\square$

Corollary 1 shows that the $\{M_n\}$ are indeed probability measures.

Corollary 1 *The $\{M_n\}$ of Definition 2 are probability measures.*

PROOF: The second expression for $\kappa(\lambda, \Lambda)$ implies that $dim(\Lambda) \geq 0$ for all Λ. The fact that $P_\Lambda \geq 0$ follows from the hypotheses on q, t and the x's, together with the skew-expansion rule (equation 7.9' on page 345 of [M]) for Macdonald polynomials. Thus $M_n(\lambda) \geq 0$ for all λ. From the definition of M_1 it is a probability measure. For larger M_n this follows from induction and the equation

$$
\begin{aligned}
1 ~=~ &\sum_{\lambda \vdash n-1} M_n(\lambda) \\
=~ &\sum_{\lambda \vdash n-1} \sum_{\Lambda:\lambda \nearrow \Lambda} \frac{dim(\lambda)}{dim(\Lambda)} \kappa(\lambda, \Lambda) M_n(\Lambda) \\
=~ &\sum_{\Lambda \vdash n} \sum_{\lambda:\lambda \nearrow \Lambda} \frac{dim(\lambda)}{dim(\Lambda)} \kappa(\lambda, \Lambda) M_n(\Lambda) \\
=~ &\sum_{\Lambda \vdash n} M_n(\Lambda).
\end{aligned}
$$

$\square$

As a consequence of the fact that the M_n are coherent, we obtain for free a method of sampling from them. This principle is implicit in the literature (e.g. page 144 of [Ke1]), but there is a surprising simplication which occurs in our examples.

Proposition 1 *Starting from $\emptyset$, at each stage move to a larger partition according to the rule that the chance of going from λ to Λ is $\frac{P_\Lambda \psi'_{\Lambda/\lambda}}{P_\lambda}$. Then after n steps the probability of being at the partition Λ is $M_n \Lambda$.*

PROOF: In general the transition probabilities from λ to Λ to sample from a coherent family $\{M_n\}$ is $\frac{\kappa(\lambda,\Lambda)M_n(\Lambda)dim(\lambda)}{M_{n-1}(\lambda)dim(\Lambda)}$. These sum to 1 by the definition of coherence, and sample from M_n because

$$\sum_{\gamma:\gamma(0)=\emptyset \nearrow \cdots \nearrow \gamma(n)=\Lambda} \prod_{j=1}^{n} \frac{\kappa(\gamma(j-1),\gamma(j))M_j(\gamma(j))dim(\gamma(j-1))}{M_{j-1}(\gamma(j-1))dim(\gamma(j))}$$

$$= M_n(\Lambda) \sum_{\gamma:\gamma(0)=\emptyset \nearrow \cdots \nearrow \gamma(n)=\Lambda} \frac{\prod_{j=1}^{n}\kappa(\gamma(j-1),\gamma(j))}{dim(\Lambda)}$$

$$= M_n(\Lambda).$$

This principle together with the formula for $M_n(\Lambda)$ inside the proof of Theorem 1, imply the proposition. $\square$

As will be seen in Section 3, in special cases the algorithm of Proposition 1 yields known results. Curiously, the transition probabilities of Proposition 1 are exactly those on page 585 of [F1], if one conditions on each coin coming up heads once. The motivating example there was the probabilistic study of the $z-1$ part of the Jordan form of a random element of $GL(n,q)$.

3 Examples

This section gives some examples of the constructions in the previous section. Before doing so, we define the Martin boundary Δ and Poisson kernel $\Phi : \Gamma \times \Delta \mapsto R$ of a branching as in [Ke1], which the reader should consult for a fuller treatment. One requires that Δ is a compact topological space and that there is a map $i : \Gamma \mapsto \Delta$ such that

1. For every $\omega \in \Delta$ the function $\phi_\Lambda(\omega) = \Phi(\Lambda,\omega)$ is harmonic with respect to the branching.

2. The functions $\Phi_\Lambda(\omega)$ are continuous and span a dense linear subspace in the space of continuous functions on Δ.

3. For every $\omega \in \Delta$, the measures $i(dim(\Lambda)\Phi(\lambda;\omega))$ converge weakly as $n \to \infty$ to the point mass δ_ω at ω.

In the case of the Young lattice, the boundary Δ is the space of pairs $(\alpha;\beta)$ such that $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$, $\beta_1 \geq \beta_2 \geq \cdots \geq 0$ and $\sum_{i=1}^{\infty}\alpha_i + \sum_{i=1}^{\infty}\beta_i \leq 1$. The map i send a partition Λ to $(\frac{f_1}{n},\frac{f_2}{n},\ldots;\frac{g_1}{n},\frac{g_2}{n},\cdots)$ where $f_i = \Lambda_i - i + \frac{1}{2}$ and $g_i = \Lambda'_i - i + \frac{1}{2}$. The Poisson kernel $\Phi(\Lambda;\alpha,\beta)$ is $s_\Lambda(\alpha;\beta;\gamma)$ where the $s_\Lambda(\alpha;\beta;\gamma)$ are the extended Schur functions defined for instance on page 147 of [Ke1].

1. **Upper triangular matrices**

Suppose that $q = 0$ and $t = \frac{1}{q}$, where this second q is the size of a finite field. Further, set $x_i = \frac{1}{q^{i-1}} - \frac{1}{q^i}$. Several simplifications take place. First, the multiplicities have a simple description; letting i be the column to which one adds in order to go from λ to Λ, it follows that $\kappa(\lambda, \Lambda) = q^{\lambda'_i} + q^{\lambda'_i - 1} + \cdots + q^{\lambda'_{i+1}}$. This is always integral. Second, $dim(\Lambda)$ reduces to a Green's polynomial $Q^{\Lambda}(q) = Q^{\Lambda}_{(1^n)}(q)$ as in Section 3.7 of [M]. These polynomials are important in the representation theory of the finite general linear groups.

The third and fourth simplifications are significant enough to be stated as propositions.

Proposition 2 *With the above specializations, $M_n(\lambda)$ is the probability that a uniformly chosen element of $T_n(\lambda)$ has Jordan form of shape λ.*

PROOF: This follows by comparison with the formula in Theorem 1 of [F2]. $\square$

Corollary 2 *([B],[Ki1]) The Jordan form of a uniformly chosen element of $T_n(\lambda)$ can be sampled from by stopping the following procedure after n steps:*

Starting with the empty partition, at each step transition from a partition λ to a partition Λ by adding a dot to column i chosen according to the rules

- *$i = 1$ with probability $\frac{1}{q^{\lambda'_1}}$*
- *$i = j > 1$ with probability $\frac{1}{q^{\lambda'_j}} - \frac{1}{q^{\lambda'_{j-1}}}$*

PROOF: This follows easily from the following five ingredients: Proposition 1, Proposition 2, homogeneity of P_Λ (which implies that $P_\Lambda(x) = (1 - \frac{1}{q})P_\Lambda(1, \frac{1}{q}, \frac{1}{q^2}, \cdots; 0, \frac{1}{q})$), Macdonald's principal specialization formula (page 337 of [M]), and a piece of paper. $\square$

Note that Borodin [B], has shown that the asymptotic Jordan form of a random element of $T(n)$ has the following shape: the longest block has size $(1 - \frac{1}{p})n$, the second block has size $(\frac{1}{p} - \frac{1}{p^2})n$, etc. This suggests to us that the harmonic function $\frac{M_n(\Lambda)}{dim(\Lambda)}$ is extremal. Is it extremal for other x_i such that $\sum x_i = 1$? This brings us to the following

Problem: Find the Martin boundary of the Bratteli diagram in this example.

2. **Schur functions**

A second example of interest occurs when $q = t < 1$. Letting i be the column to which one adds in order to go from λ to Λ, it is not hard to rewrite $\kappa(\lambda, \Lambda)$ as

$$\frac{1}{q^{\lambda'_i}} \frac{\prod_{s \in \lambda}[h_\Lambda(s)]}{\prod_{s \in \lambda}[h_\lambda(s)]}.$$

One checks (using the fact that f^Λ is the number of paths in the Young lattice from $\emptyset$ to Λ and that the product of multiplicities is path independent) that the dimension also has a nice simplification, namely $dim(\Lambda) = \frac{f^\Lambda \prod_{s \in \Lambda} h[s]}{q^{n(\Lambda)}}$.

The measure $M_n(\Lambda)$ reduces to $s_\Lambda f^\Lambda$, where s_Λ is a Schur function. Setting $x_1 = \cdots = x_n = \frac{1}{n}$ and letting $n \to \infty$, one obtains Plancharel measure, which is important in representation theory and random matrix theory. A method for sampling from it was found in [GNW] (see [Ke3] for extensions).

Letting $x_1 = \cdots = x_n$ satisfy $\sum x_i = 1$ (all other $x_j = 0$) gives a natural deformation of Plancharel measure, studied for instance by [ITW]. Stanley [S] shows that this measure on partitions also arises by applying the RSK algorithm to a random permutation distributed after a biased riffle shuffle. Since the quantities in Proposition 1 have simple expressions under this specialization (e.g. page 45 of [M]), the sampling algorithm is useful.

Theorem 2 *The Martin boundary in these examples is the same as for the Young lattice.*

PROOF: We use the ergodic method (Section 8 of [Ke1]). The map i is the same as for the Young lattice and the Poisson kernel is defined as $\frac{q^{n(\Lambda)} s_\lambda(\alpha,\beta;\gamma)}{\prod_{s\in\Lambda}[h_\Lambda(s)]}$. To see that the first condition of a boundary is met, recall that the function $\phi_t(\Lambda) = \lim_{n\to\infty} \frac{dim(\Lambda,\nu_n)}{dim(\emptyset,\nu_n)}$ is harmonic if it exists (here ν_n is a sequence of vertices of a path with each $\nu_n \in \Gamma_n$). In fact it is true that

$$\frac{dim(\Lambda,\nu)}{dim(\emptyset,\nu)} = \frac{q^{n(\Lambda)}}{\prod_{s\in\Lambda}[h_\Lambda]} \frac{dim^*(\nu-\Lambda)}{dim^*(\nu)},$$

where dim^* denotes dimension in the Young lattice and $dim^*(\nu - \Lambda)$ is the number of paths in the Young lattice from Λ to ν. This follows from the observation that the product of the mulitplicities κ along a path in the Bratteli diagram of this example depends only on the endpoints of the path. The fact that the second condition of a boundary is met follows from a generating function argument showing that the $\Phi_\Lambda(\omega)$ separate points of the boundary and the Stone-Wierstrass theorem. The third condition of a boundary amounts to exactly the same condition as for the Young lattice, and thus holds. $\square$

3. **Jack symmetric functions**

A third example of interest occurs by setting $q = \frac{1}{t^\theta}$, and taking the limit as $t \to 1$. The multiplicity function $\kappa(\lambda, \Lambda)$ takes the form

$$\prod_{s\in R_{\Lambda/\lambda}} \frac{a_\Lambda(s) + 1 + \theta l_\Lambda(s)}{a_\lambda(s) + 1 + \theta l_\lambda(s)} \prod_{s\in C_{\Lambda/\lambda}} \frac{a_\Lambda(s) + \theta(l_\Lambda(s) + 1)}{a_\lambda(s) + \theta(l_\lambda(s) + 1)}.$$

We do not know of a simple general expression for $dim(\Lambda)$.

Setting $\theta = 0$ (Kingman branching) leads to trouble with our formulation as it amount to setting $q = 1$ before taking the limit $t \to 1$. Setting $\theta = 1$ (Schur functions) has already been considered. The case of zonal functions (i.e. $\theta = \frac{1}{2}$) merits further investigation.

4   Acknowledgments

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